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Derivation of the fourth-order tangent operator based on a generalized eigenvalue problem

P. Betsch, P. Steinmann*

Lehrstuhl für Technische Mechanik, Universität Kaiserslautern, Germany

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Abstract

Continuous and algorithmic forms of the fourth-order tangent operator corresponding to isotropic multiplicative elasto-plasticity are derived by generalizing an approach originally developed for finite elasticity. The Lagrangian description of large-strain elasto-plasticity leads to a generalized eigenvalue problem which facilitates certain tensor representations with respect to a reciprocal set of left and right eigenvectors. The tangent operators take an extremely simple form due to the resolution in the basis spanned by the right eigenvectors. Remarkably, these new developments reveal that the algorithmic version of the tangent operator preserves the structure of the continuous counterpart.

1. Introduction

Our present contribution generalizes previously published derivatives of isotropic tensor functions due to Bowen and Wang (1970) and Chadwick and Ogden (1971). Both works contain a very elegant derivation of the fourth-order tangent operator which represents the gradient of the second Piola–Kirchhoff stress tensor with respect to the Green–Lagrangian strain tensor. The method described in these references has been developed for large-strain isotropic *elasticity* and is essentially based on a *special* eigenvalue problem which determines the principal stretches as well as an *orthonormal* set of eigenvectors. The calculation of the eigenvectors is characteristic for this approach and leads to extremely simple expressions for the tangent operator. A corresponding numerical implementation within a finite element formulation is described in Reese and Wriggers (1995). An algorithmic version of multiplicative elasto-plasticity is treated in Wriggers et al. (1996) where again, within a spatial setting, a special eigenvalue problem appears.

The inherent simplicity of the aforementioned approach becomes evident when compared with an

* Corresponding author. Tel.: 00 49 0631 205 2419; fax: 00 49 0631 205 2128.
E-mail address: ps@rhrk.uni-kl.de (P. Steinmann)

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alternative derivation based on the first representation theorem for isotropic tensor functions (see e.g. Morman, 1986; Gurtin, 1981). The alternative derivation is described in Simo and Taylor (1991) for the case of finite elasticity, a more compact formulation can be found in Miehe (1993). A related finite element implementation of the second approach can be found in Simo and Taylor (1991) for the case of finite elasticity, while an algorithmic version of large-strain elasto-plasticity is treated in Simo (1992), again within a spatial setting.

Our goal is to derive the *continuous* as well as the *algorithmic* form of the fourth-order tangent operator for a *Lagrangian* formulation of large-strain elasto-plasticity. As pointed out by Ibrahimbegovic (1994) the material setting is appropriate for the description of large inelastic deformations of space-curved membrane shells. Moreover, recently developed shell finite elements for large-strain deformations rely on the assumed strain method which intrinsically is a material formulation (Betsch and Stein, 1996; and the references therein). Accordingly, the algorithmic version of the tangent operator in the material setting we aim for is especially useful for the implementation into assumed strain elements, particularly shell finite elements.

The *material* version of large-strain multiplicative elasto-plasticity implies a *generalized* eigenvalue problem (Ibrahimbegovic, 1994; Betsch and Stein, 1998). It turns out that the generalized eigenvalue problem belongs to the class of symmetric-definite eigenproblems which makes possible certain tensor representations with respect to a reciprocal set of left and right eigenvectors. These representations are the cornerstone for the derivation of both the continuous and algorithmic versions of the continuous fourth-order tangent operator. We show that the simplicity and elegance of the original approach due to Bowen and Wang (1970) and Chadwick and Ogden (1971) can be entirely preserved in the more general context considered here.

After a short summary of the underlying constitutive model of multiplicative elasto-plasticity formulated within the Lagrangian setting, in Section 2.2 we rephrase the formulation by taking into account the special structure of the generalized eigenvalue problem. In Section 2.3 we present the derivation of the fourth-order tangent operator. Section 2.4 is devoted to the algorithmic counterpart of the continuous description considered before. Finally, in Section 2.5 we consider the case of finite elasticity which is contained as special case in the generalized setting at hand.

2. Derivation of the tangent operators

2.1. Outline of the problem

Within a material setting of large-strain isotropic elasto plasticity our goal is to derive the fourth-order tangent operator $\mathbb{L} = 2\partial_C \boldsymbol{S}$ such that the rate equation $\dot{\boldsymbol{S}} = \mathbb{L} : \frac{1}{2}\dot{\boldsymbol{C}}$ holds. Here, $\boldsymbol{C}$ is the symmetric second-order *deformation tensor* (right Cauchy–Green tensor), $\boldsymbol{S}$ is the symmetric second-order *second Piola–Kirchhoff stress tensor* and a superposed dot denotes the material time derivative. Following Ibrahimbegovic (1994), the material version of multiplicative elasto-plasticity leads to the generalized eigenvalue problem

$$\left[\boldsymbol{C} - \lambda_A^2 \boldsymbol{C}^p\right]\boldsymbol{N}^A = \boldsymbol{0}, \tag{1}$$

where the symmetric second-order tensor $\boldsymbol{C}^p$ takes into account inelastic deformations. Since $\boldsymbol{C} \in \mathbb{R}^{3\times 3}$ is symmetric and $\boldsymbol{C}^p \in \mathbb{R}^{3\times 3}$ is symmetric positive definite, the eigenvalue problem (1) belongs to the class of symmetric-definite generalized eigenproblems. The set of generalized eigenvalues $\lambda^2(\boldsymbol{C}, \boldsymbol{C}^p) = \{\lambda_1^2, \lambda_2^2, \lambda_3^2\}$ is given by $\lambda^2(\boldsymbol{C}, \boldsymbol{C}^p) = \{\lambda^2 | \det[\boldsymbol{C} - \lambda^2 \boldsymbol{C}^p] = 0\}$. Speaking in physical terms, λ represents the principal *elastic* stretch (see also Remark 2.1).

Assuming the existence of a strain energy function given as symmetric function of the principal elastic stretches, i.e. $\Psi = \Psi(\lambda_1, \lambda_2, \lambda_3)$, the second Piola–Kirchhoff stress tensor can be calculated via

$$S = 2\partial_C \Psi . \tag{2}$$

Under the restriction of isotropy we further consider a yield function $\Phi = \Phi(\tau_1, \tau_2, \tau_3)$, given as symmetric function of the principal Kirchhoff-stresses $\tau_A = \lambda_A \partial_{\lambda_A} \Phi, (A = 1, 2, 3)$. The evolution of plastic deformation may now be written in the form (Ibrahimbegovic, 1994)

$$\partial_t C^{p-1} = -2\dot{\gamma} C^{-1} \partial_S \Phi C^{p-1}, \tag{3}$$

which conforms with the principle of maximum plastic dissipation.

Remark 2.1. *The considered isotropic theory of large-strain elasto-plasticity relies on a multiplicative decomposition of the deformation gradient in the form*

$$\boldsymbol{F} = \boldsymbol{F}^e \boldsymbol{F}^p, \tag{4}$$

where, $\boldsymbol{F}^e$ and $\boldsymbol{F}^p$ represent the elastic and plastic part, respectively. We refer to Lubliner (1990) or Maugin (1992) for further background material on multiplicative elasto-plasticity. The spatial counterpart of the constitutive description in the reference configuration summarized above involves the symmetric eigenvalue problem

$$\left[\boldsymbol{b}^e \boldsymbol{g} - \lambda_A^2 \mathbf{I}\right] \boldsymbol{n}_A = \mathbf{0} \tag{5}$$

where $\boldsymbol{g}$ is the covariant metric tensor in the current configuration and $\boldsymbol{b}^e = \boldsymbol{F}^e \boldsymbol{F}^{e^T}$ is the elastic left Cauchy–Green tensor. Similarly, in eqn (1) we have $\boldsymbol{C}^p = \boldsymbol{F}^{p^T} \boldsymbol{F}^p$, the plastic right Cauchy–Green tensor. Furthermore, in eqn (5), λ_A are the principal elastic stretches. Due to the special structure of the eigenvalue problem, eqn (5), there is an orthonormal basis consisting of the eigenvectors $\boldsymbol{n}_A$. Both eigenvalue problem, eqn (1) and (5), are connected by a similarity transformation which preserves the eigenvalues. The transformation can be easily performed by taking into account the identities $\boldsymbol{C} = \boldsymbol{F}^T \boldsymbol{g}^{\boldsymbol{F}}$ and $\boldsymbol{C}^{p-1} = \boldsymbol{F}^{-1} \boldsymbol{b}^e \boldsymbol{F}^{-T}$, such that $\boldsymbol{F}^{-1} \boldsymbol{b}^e \boldsymbol{g}^{\boldsymbol{F}} = \boldsymbol{C}^{p-1} \boldsymbol{C}$. Consequently, the similarity transformation leads to the relation $\boldsymbol{N}^A = \boldsymbol{F}^{-1} \boldsymbol{n}_A$.

We further remark that alternative equivalent versions of the flow rule (3), formulated with respect to different configurations, have been derived e.g. by Miehe and Stein (1992), Simo (1992) and Hackl (1997).

2.2. Formulation based on generalized eigenvalue problem

The key to our derivation of the material fourth-order tangent tensor is the resolution of the tensor quantities based on the reciprocal set of left and right eigenvectors associated with the generalized eigenvalue problem, eqn (1). First, let us introduce the reciprocal set of left and right eigenvectors corresponding to the eigenvalue problem eqn (1), which may be written in the form

$$[\boldsymbol{C}^{p-1} \boldsymbol{C}] \boldsymbol{N}^A = \boldsymbol{\lambda}_A^2 \boldsymbol{N}^A, \tag{6}$$

such that $\boldsymbol{N}^A$ are the right eigenvectors of $[\boldsymbol{C}^{p-1} \boldsymbol{C}]$ corresponding to the eigenvalues λ_A^2. We further introduce

$$\boldsymbol{N}_A = \boldsymbol{C} \boldsymbol{N}^A, \tag{7}$$

where $\boldsymbol{N}_A$ are the left eigenvectors of $[\boldsymbol{C}^{p-1}\boldsymbol{C}]$, since eqn (1) implies the relation

$$\boldsymbol{N}_A \cdot \left[\boldsymbol{C}^{p-1}\boldsymbol{C}\right] = \lambda_A^2 \boldsymbol{N}_A. \tag{8}$$

Scalar-multiplication of eqn (6) by $\boldsymbol{N}_A$ yields $\boldsymbol{N}_A \cdot [\boldsymbol{C}^{p-1}\boldsymbol{C}]\boldsymbol{N}^B = \lambda_B^2 \boldsymbol{N}_A \cdot \boldsymbol{N}^B$. Employing eqn (8), this equation may be written in the form

$$\left[\lambda_A^2 - \lambda_B^2\right]\boldsymbol{N}_A \cdot \boldsymbol{N}^B = 0. \tag{9}$$

We further introduce the normalization

$$\boldsymbol{N}^A \cdot \boldsymbol{C}\,\boldsymbol{N}^A = 1, \tag{10}$$

such that the orthogonality property

$$\boldsymbol{N}_A \cdot \boldsymbol{N}^B = \delta_A^B \tag{11}$$

holds, where δ_A^B is the Kronecker delta. Note that the basis spanned by $\boldsymbol{N}^A$ and $\boldsymbol{N}_A$ is in general not orthonormal or even orthogonal.

Next, we elaborate on special representations for $\boldsymbol{C}$ and $\boldsymbol{C}^p$ which are essential for our further developments. These representations are possible due to the fact that the eigenvalue problem under consideration belongs to the class of so-called *symmetric-definite generalized eigenvalue problems*. We refer to Golub and Van Loan (1996), section 8.7, or Ericksen (1960), section 37, for extensive investigations of symmetric-definite generalized eigenvalue problems. Accordingly, in the present context, the following representations are possible:

$$\boldsymbol{C} = \sum_{A=1}^{3} \boldsymbol{N}_A \otimes \boldsymbol{N}_A \quad \text{and} \quad \boldsymbol{C}^p = \sum_{A=1}^{3} \frac{1}{\lambda_A^2} \boldsymbol{N}_A \otimes \boldsymbol{N}_A. \tag{12}$$

Furthermore,

$$\boldsymbol{C}^{-1} = \sum_{A=1}^{3} \boldsymbol{N}^A \otimes \boldsymbol{N}^A \quad \text{and} \quad \boldsymbol{C}^{p-1} = \sum_{A=1}^{3} \lambda_A^2 \boldsymbol{N}^A \otimes \boldsymbol{N}^A. \tag{13}$$

In addition to that, the tensor $\boldsymbol{C}^{p-1}\boldsymbol{C}$ can be expressed uniquely in terms of its eigenvalues and a reciprocal set of left and right eigenvectors. Accordingly, in the present case we obtain

$$\boldsymbol{C}^{p-1}\boldsymbol{C} = \sum_{A=1}^{3} \lambda_A^2 \boldsymbol{N}^A \otimes \boldsymbol{N}_A. \tag{14}$$

Note further, that the identity tensor may be written in the form

$$\mathbf{I} = \sum_{A=1}^{3} \boldsymbol{N}^A \otimes \boldsymbol{N}_A. \tag{15}$$

Remark 2.2. *We emphasize again that the vectors $\boldsymbol{N}^A$ as well as $\boldsymbol{N}_A$ are in general not mutually orthogonal. Therefore, eqn (12) should not be confused with the spectral decomposition of* $\boldsymbol{C}$ and $\boldsymbol{C}^p$. *These expressions are merely component representations of* $\boldsymbol{C}$ and $\boldsymbol{C}^p$ *with respect to the (skew) vectors $\boldsymbol{N}_A$. Accordingly, the corresponding matrix representations of* $\boldsymbol{C}$ and $\boldsymbol{C}^p$ *take the particularly simple diagonal forms*

$$\left[C_{ij}\right] = \text{ diag }(1, 1, 1) \quad \text{and} \quad \left[C_{ij}^{p}\right] = \text{ diag }\left(\lambda_1^{-2}, \lambda_2^{-2}, \lambda_3^{-2}\right) \tag{16}$$

in the basis consisting of vectors $\boldsymbol{N}_A$. These results are due to the fact that symmetric-definite pencils, like $\boldsymbol{C} - \lambda^2 \boldsymbol{C}^p$ in the present case, can be simultaneously diagonalized.

Now we are in a position to derive preliminary results which will be needed for the derivation of the tangent operators. The material time derivative of eqn $(12)_1$ yields

$$\dot{\boldsymbol{C}} = \sum_{A=1}^{3} \left[\dot{\boldsymbol{N}}_A \otimes \boldsymbol{N}_A + \boldsymbol{N}_A \otimes \dot{\boldsymbol{N}}_A\right], \tag{17}$$

which may also be written in the form $\dot{\boldsymbol{C}} = (\dot{\boldsymbol{C}})^{ij} \boldsymbol{N}_i \otimes \boldsymbol{N}_j$ where $(\dot{\boldsymbol{C}})^{AB} = \boldsymbol{N}^A \cdot \dot{\boldsymbol{C}} \boldsymbol{N}^B$ are the components of $\dot{\boldsymbol{C}}$ with respect to the basis $\{\boldsymbol{N}_A\}$, given by $(\dot{\boldsymbol{C}})^{AB} = \boldsymbol{N}^A \cdot \dot{\boldsymbol{N}}_B + \boldsymbol{N}^B \cdot \dot{\boldsymbol{N}}_A$. Here and in the remainder, the summation convention applies to repeated lower case letters, whereas repeated upper case letters are not assumed summed unless indicated by a summation symbol.

Now, $\dot{\boldsymbol{C}}$ may also be represented by

$$(\dot{\boldsymbol{C}})^{AA} = 2\boldsymbol{N}^A \cdot \dot{\boldsymbol{N}}_A \quad A = 1, 2, 3,$$

$$(\dot{\boldsymbol{C}})^{AB} = \boldsymbol{N}^A \cdot \dot{\boldsymbol{N}}_B + \boldsymbol{N}^B \cdot \dot{\boldsymbol{N}}_A \quad A \neq B. \tag{18}$$

Similarly, we consider $\dot{\boldsymbol{C}}^p = (\dot{\boldsymbol{C}}^p)^{ij} \boldsymbol{N}_i \otimes \boldsymbol{N}_j$, such that time differentiation of eqn $(12)_2$ leads to the component representations

$$\left(\dot{\boldsymbol{C}}^p\right)^{AA} = \frac{2}{\lambda_A^2}\left[\boldsymbol{N}^A \cdot \dot{\boldsymbol{N}}_A - \frac{\dot{\lambda}_A}{\lambda_A}\right] \quad A = 1, 2, 3,$$

$$\left(\dot{\boldsymbol{C}}^p\right)^{AB} = \frac{1}{\lambda_B^2} \boldsymbol{N}^A \cdot \dot{\boldsymbol{N}}_B + \frac{1}{\lambda_A^2} \boldsymbol{N}^B \cdot \dot{\boldsymbol{N}}_A \quad A \neq B. \tag{19}$$

Proposition 2.1. *With respect to the basis $\{\boldsymbol{N}^A\}$ the second Piola–Kirchhoff stress tensor (2) takes the form*

$$S = \sum_{A=1}^{3} \tau_A \boldsymbol{N}^A \otimes \boldsymbol{N}^A, \tag{20}$$

with components $\tau_A = \lambda_A \partial_{\lambda_A} \Psi$.

Proof. From eqn (2), by the chain rule we have

$$\boldsymbol{S} = 2 \sum_{A=1}^{3} \partial_{\lambda_A} \Psi \partial_{\boldsymbol{C}} \lambda_A, \tag{21}$$

where the term $\partial_C \lambda_A$ may be calculated by employing eqn $(18)_1$, i.e. $\boldsymbol{N}^A \cdot \dot{\boldsymbol{C}} \boldsymbol{N}^A = 2\boldsymbol{N}^A \cdot \dot{\boldsymbol{N}}_A$, together with eqn $(19)_1$, i.e. $\boldsymbol{N}^A \cdot \dot{\boldsymbol{C}}^p \boldsymbol{N}^A = 2/\lambda_A^2 \left[\boldsymbol{N}^A \cdot \dot{\boldsymbol{N}}_A - \dot{\lambda}_A/\lambda_A\right]$. Accordingly,

$$\dot{\lambda}_A = \frac{\lambda_A}{2} \boldsymbol{N}^A \cdot \dot{\boldsymbol{C}} \boldsymbol{N}^A - \frac{\lambda_A^3}{2} \boldsymbol{N}^A \cdot \dot{\boldsymbol{C}}^p \boldsymbol{N}^A, \tag{22}$$

which implies

$$\partial_C \lambda_A = \frac{\lambda_A}{2} \boldsymbol{N}^A \otimes \boldsymbol{N}^A. \tag{23}$$

Proposition 2.2. *The resolution of the flow rule (3) with respect to the left eigenvectors* $\{\boldsymbol{N}_A\}$ *takes the form*

$$\dot{\boldsymbol{C}}^p = 2\dot{\gamma} \sum_{A=1}^{3} \frac{\partial_{\tau_A} \Phi}{\lambda_A^2} \boldsymbol{N}_A \otimes \boldsymbol{N}_A. \tag{24}$$

Proof. First, we show that

$$\partial_S \tau_A = \boldsymbol{N}_A \otimes \boldsymbol{N}_A. \tag{25}$$

Material time differentiation of eqn (20) leads to $\boldsymbol{N}_A \cdot \dot{\boldsymbol{S}} \boldsymbol{N}_A = \dot{\tau}_A - 2\tau_A \boldsymbol{N}^A \cdot \dot{\boldsymbol{N}}_A$. In view of eqn $(18)_1$ we have $2\boldsymbol{N}^A \cdot \dot{\boldsymbol{N}}_A = \boldsymbol{N}^A \cdot \dot{\boldsymbol{C}} \boldsymbol{N}^A$, such that

$$\dot{\tau}_A = \boldsymbol{N}_A \cdot \dot{\boldsymbol{S}} \boldsymbol{N}_A + \tau_A \boldsymbol{N}^A \cdot \dot{\boldsymbol{C}} \boldsymbol{N}^A, \tag{26}$$

which implies eqn (25). Now, the particular form, eqn (24), of the flow rule can be obtained from eqn (3) by using the identity $\dot{\boldsymbol{C}}^p = -\boldsymbol{C}^p \partial_t \boldsymbol{C}^{p-1} \boldsymbol{C}^p$ in combination with the spectral representations (13) associated with the generalized eigenvalue problem.

Next, we compare the expression (19) for $\dot{\boldsymbol{C}}^p$, emanating from the generalized eigenvalue problem, with the representation (24) of the continuous flow rule. Accordingly, we obtain the following relations which play an important role in our subsequent developments:

$$\begin{aligned} \frac{1}{2}(\dot{\boldsymbol{C}})^{AA} &= \frac{\dot{\lambda}_A}{\lambda_A} + \dot{\gamma}\, \partial_{\tau_A} \Phi \quad A = 1, 2, 3, \\ \lambda_A^2 \boldsymbol{N}^A \cdot \dot{\boldsymbol{N}}_B + \lambda_B^2 \boldsymbol{N}^B \cdot \dot{\boldsymbol{N}}_A &= 0 \quad A \neq B. \end{aligned} \tag{27}$$

2.3. Continuous elasto-plastic tangent operator

We aim at the derivation of the fourth-order tangent operator $\mathbb{L} = 2\partial_C \boldsymbol{S}$ corresponding to the material formulation of large-strain elasto-plasticity described above. To this end we choose the representation

$$\mathbb{L} = \boldsymbol{L}_{ijkl} \boldsymbol{N}^i \otimes \boldsymbol{N}^j \otimes \boldsymbol{N}^k \otimes \boldsymbol{N}^l, \tag{28}$$

with components corresponding to the right eigenvectors $\boldsymbol{N}^A$ of the generalized eigenvalue problem, eqn (1).

Material time differentiation of the second Piola–Kirchhoff stress tensor in the form of eqn (20) yields the components

$$\begin{aligned} (\dot{\boldsymbol{S}})_{AA} &= \dot{\tau}_A - 2\tau_A \boldsymbol{N}^A \cdot \dot{\boldsymbol{N}}_A \quad A = 1, 2, 3, \\ (\dot{\boldsymbol{S}})_{AB} &= -\left[\tau_A \boldsymbol{N}^A \cdot \dot{\boldsymbol{N}}_B + \tau_B \boldsymbol{N}^B \cdot \dot{\boldsymbol{N}}_A\right] \quad A \neq B, \end{aligned} \tag{29}$$

such that $\dot{\boldsymbol{S}} = (\dot{\boldsymbol{S}})_{ij} \boldsymbol{N}^i \otimes \boldsymbol{N}^j$. Using eqn $(27)_2$ in conjunction with eqn $(18)_1$ and the continuous elasto-plastic moduli D_{AB}^{ep} derived in Appendix A, we arrive at

$$(\dot{\boldsymbol{S}})_{AA} = \sum_{B=1}^{3} \left[D^{ep}_{AB} - 2\tau_A\delta_{AB}\right]\frac{1}{2}(\dot{\boldsymbol{C}})^{BB} \quad A = 1, 2, 3,$$

$$(\dot{\boldsymbol{S}})_{AB} = \frac{1}{\lambda_B^2}\left[\tau_B\lambda_A^2 - \tau_A\lambda_B^2\right]\boldsymbol{N}^A \cdot \dot{\boldsymbol{N}}_B \quad A \neq B. \tag{30}$$

Furthermore, making use of eqn $(27)_2$, the components of $\dot{\boldsymbol{C}}$ given in the form of eqn (18) may now be written as

$$(\dot{\boldsymbol{C}})^{AA} = 2\boldsymbol{N}^A \cdot \dot{\boldsymbol{N}}_A \quad A = 1, 2, 3,$$

$$(\dot{\boldsymbol{C}})^{AB} = \frac{1}{\lambda_B^2}\left[\lambda_B^2 - \lambda_A^2\right]\boldsymbol{N}^A \cdot \dot{\boldsymbol{N}}_B \quad A \neq B. \tag{31}$$

Substituting from eqn $(31)_2$ into eqn $(30)_2$ leads to

$$(\dot{\boldsymbol{S}})_{AB} = \frac{\tau_B\lambda_A^2 - \tau_A\lambda_B^2}{\lambda_B^2 - \lambda_A^2}\left(\dot{\boldsymbol{C}}^{AB}\right) \quad A \neq B. \tag{32}$$

Since $\dot{\boldsymbol{S}} = \mathbb{L} : \frac{1}{2}\dot{\boldsymbol{C}}$ and with regard to eqn (28), the component equation

$$(\dot{\boldsymbol{S}})_{AB} = \sum_{I=1}^{3} L_{ABII}\frac{1}{2}(\dot{\boldsymbol{C}})^{II} + \sum_{I\neq J} L_{ABIJ}\frac{1}{2}(\dot{\boldsymbol{C}})^{IJ} \tag{33}$$

has to hold for *arbitrary* rates $(\dot{\boldsymbol{C}})^{IJ} \in \mathbb{R}^6$. Now, the components L_{ABCD} can be determined by inserting eqns $(30)_1$ and eqns (32) into eqn (33).

First, let us consider the case $A = B$ in eqn (33). In view of eqn $(30)_1$ and the arbitrariness of $(\dot{\boldsymbol{C}})^{AB} \in \mathbb{R}^6$, we immediately obtain the result

$$L_{AAII} = D^{ep}_{AI} - 2\tau_A\delta_{AI}. \tag{34}$$

In the case $A \neq B$, eqn (32) in conjunction with eqn (33) leads to

$$L_{ABAB} = \frac{\tau_B\lambda_A^2 - \tau_A\lambda_B^2}{\lambda_B^2 - \lambda_A^2} \quad A \neq B, \tag{35}$$

where the minor symmetry of $\mathbb{L}$, i.e. $L_{ABAB} = L_{ABBA}$, has been taken into account. All components not specified in eqns (34) and (35) vanish identically. To the authors' knowledge, the particular representations (34) and (35) of the *continuous* tangent operator have not been derived in the literature before. As will be shown in the next section, the algorithmic version of the tangent operator inherits completely the simple structure of its continuous counterpart.

2.3.1. Coalescent eigenvalues

The foregoing derivation of the fourth-order tangent operator relies on the implicit assumption of distinct eigenvalues. The case of multiple eigenvalues can be incorporated by means of a limiting process in a similar way as was proposed by Bowen and Wang (1970) and Chadwick and Ogden (1971) for elasticity. Accordingly, when λ_A and λ_B coincide, (35) needs to be replaced by

$$L_{ABAB} = \tfrac{1}{2}[D_{BB} - D_{AB}] - \tau_A \quad \text{if } \lambda_A = \lambda_B,\ A \neq B. \tag{36}$$

Here, D_{AB} are the components in eqn (A2).

2.4. Algorithmic elasto-plastic tangent operator

The algorithmic counterpart of the constitutive formulation described above is essentially based on the integration of the flow rule in conjunction with an operator split method, consisting of a 'trial' elastic predictor followed by a plastic corrector (Simo, 1992; and references therein). As has been shown by Betsch and Stein (1998), within the material setting of multiplicative elasto-plasticity, one obtains the generalized eigenvalue problem

$$\left[\boldsymbol{C}_{n+1} - \lambda^2_{A^*}\boldsymbol{C}^p_n\right]\boldsymbol{N}^A* = 0, \tag{37}$$

corresponding to the trial elastic step. Here, $\boldsymbol{C}^p_n$ is the plastic history field belonging to the last load step, whereas $\boldsymbol{C}_{n+1}$ is the deformation tensor of a new configuration calculated by means of an iterative solution procedure, e.g. the Newton method.

According to our previous investigations, the symmetric-definite eigenproblem (37) gives rise to the following representations:

$$\boldsymbol{C}_{n+1} = \sum_{A=1}^{3} \boldsymbol{N}_{A^*} \otimes \boldsymbol{N}_{A^*} \quad \text{and} \quad \boldsymbol{C}^p_n = \sum_{A=1}^{3} \frac{1}{\lambda^2_{A^*}} \boldsymbol{N}_{A^*} \otimes \boldsymbol{N}_{A^*}. \tag{38}$$

These expressions are analogous to eqn (12) and correspond to the symmetric-definite eigenproblem (37) associated with the trial elastic step.

As has been originally proposed by Weber and Anand (1990), the integration of the flow rule may be performed by means of an implicit method by employing the exponential map. In the present case, the flow rule (24) may be written in the form

$$\dot{\boldsymbol{C}}^p = 2\dot{\gamma} \sum_{A=1}^{3} \partial_{\tau_A}\Phi \boldsymbol{N}_A \otimes \boldsymbol{N}^A \boldsymbol{C}^p, \tag{39}$$

such that the algorithmic version of the flow rule follows as

$$\boldsymbol{C}^p_{n+1} = \Xi^{-1}_{\Delta}\boldsymbol{C}^p_n. \tag{40}$$

In eqn (40), the incremental integration operator is given by $\Xi^{-1}_{\Delta} = \exp[\Sigma_A 2\gamma\partial_{\tau_A}\Phi\boldsymbol{N}_A \otimes \boldsymbol{N}^A]|_{n+1}$, such that $\Xi_\Delta = \Sigma_A \exp[-2\gamma\partial_{T_A}\Phi]\boldsymbol{N}_A \otimes N^A|_{n+1}$. The algorithmic flow rule (40) may also be written in the form $\Xi_\Delta \boldsymbol{C}^p_{n+1} = \boldsymbol{C}^p_n$, so that using eqn (12) for $\boldsymbol{C}^p_{n+1}$, and eqn (38) for $\boldsymbol{C}^p_n$, yields the integrated version of the flow rule in the form

$$\varepsilon_{An+1} = \varepsilon_{A^*} - \gamma\partial_{\tau\ A}\Phi|_{n+1} \quad A = 1, 2, 3, \tag{41}$$

where, by definition $\varepsilon_A = \ln \lambda_A$. Moreover, we obtain $\boldsymbol{N}_{An+1} = \boldsymbol{N}_{A^*}$. Eqn (41) can be considered as the algorithmic counterpart of the continuous flow rule in the form of eqn $(27)_1$.

In analogy with eqn (18), within the algorithmic formulation, we may express the components of the linearized deformation tensor $\Delta\boldsymbol{C}_{n+1}$ with respect to the basis $\{\boldsymbol{N}_{An+1}\}$. For the sake of a compact notation we will omit the subscript '$n + 1$' in the remainder of this section. From eqn $(38)_1$ we calculate the components

$$(\Delta \boldsymbol{C})^{AA} = 2\boldsymbol{N}^A \cdot \Delta \boldsymbol{N}_A \quad A = 1, 2, 3,$$

$$(\Delta \boldsymbol{C})^{AB} = \boldsymbol{N}^A \cdot \Delta \boldsymbol{N}_B + \boldsymbol{N}^B \cdot \Delta \boldsymbol{N}_A \quad A \neq B. \tag{42}$$

Since $\boldsymbol{C}_n^p$ represents the converged plastic field of the last increment, $\Delta \boldsymbol{C}_n^p = \boldsymbol{0}$, such that eqn $(38)_2$ leads to the component equations

$$\begin{aligned} \boldsymbol{N}^A \cdot \Delta \boldsymbol{N}_A - \frac{\Delta \lambda_{A^*}}{\lambda_{A^*}} = 0 \quad A = 1, 2, 3, \\ \lambda_{A^*}^2 \boldsymbol{N}^A \cdot \Delta \boldsymbol{N}_B + \lambda_{B^*}^2 \boldsymbol{N}^B \cdot \Delta \boldsymbol{N}_A = 0 \quad A \neq B. \end{aligned} \tag{43}$$

Thus, the components of $\Delta \boldsymbol{C}$ in eqn (42) can also be written as

$$(\Delta \boldsymbol{C})^{AA} = 2\frac{\Delta \lambda_{A^*}}{\lambda_{A^*}} = 2\Delta \varepsilon_{A^*} \quad A = 1, 2, 3,$$

$$(\Delta \boldsymbol{C})^{AB} = \frac{1}{\lambda_{B^*}^2}\left[\lambda_{B^*}^2 - \lambda_{A^*}^2\right]\boldsymbol{N}^A \cdot \Delta \boldsymbol{N}_B \quad A \neq B, \tag{44}$$

which can be considered as the algorithmic counterpart of eqn (31). The linearization of the second Piola–Kirchhoff stress tensor can be performed along lines of the continuous case elaborated above. Accordingly, we obtain the component expressions

$$(\Delta \boldsymbol{S})_{AA} = \sum_{B=1}^{3}\left[\tilde{D}_{AB}^{ep} - 2\tau_A \delta_{AB}\right]\frac{1}{2}(\Delta \boldsymbol{C})^{BB} \quad A = 1, 2, 3,$$

$$(\Delta \boldsymbol{S})_{AB} = \frac{\tau_B \lambda_{A^*}^2 - \tau_A \lambda_{B^*}^2}{\lambda_{B^*}^2 - \lambda_{A^*}^2}(\Delta \boldsymbol{C}^{AB}) \quad A \neq B. \tag{45}$$

which are the algorithmic versions of eqns $(30)_1$ and eqn (32), respectively. In eqn (45), $\tilde{D}_{AB}^{ep}$ now represents the algorithmic elasto-plastic moduli derived in Appendix B. Eventually, as in the continuous case, we obtain the algorithmic fourth-order tangent operator in the form $\tilde{\mathbb{L}} = \tilde{L}_{ijkl}\boldsymbol{N}^i \otimes \boldsymbol{N}^j \otimes \boldsymbol{N}^k \otimes \boldsymbol{N}^l$, with components given by

$$\tilde{L}_{AABB} = \tilde{D}_{AB}^{ep} - 2\tau_A \delta_{AB} \quad A, B = 1, 2, 3,$$

$$\tilde{L}_{ABAB} = \frac{\tau_B \lambda_{A^*}^2 - \tau_A \lambda_{B^*}^2}{\lambda_{B^*}^2 - \lambda_{A^*}^2} \quad A \neq B. \tag{46}$$

such that $\Delta \boldsymbol{S} = \tilde{\mathbb{L}} : \Delta \boldsymbol{C}$. All the components not specified in eqn (46) vanish identically, i.e. there remain only nine independent components $\tilde{L}_{ABCD}$.

It is worthwhile noting that the algorithmic tangent operator inherits the structure of the continuous tangent operator which becomes evident by comparing eqn (46) with eqns (34) and (35), respectively. Accordingly, similar to the infinitesimal theory (Simo and Taylor, 1985), in the algorithmic setting the continuous elasto-plastic moduli D_{AB}^{ep} simply have to be replaced by the algorithmic elasto-plastic moduli $\tilde{D}_{AB}^{ep}$. In addition to that, in eqn $(46)_2$ the principal elastic stretch is associated with the trial elastic state.

Obviously, in the case of an elastic deformation increment, that is for $\boldsymbol{C}^p_{n+1} = \boldsymbol{C}^p_n$, the continuous and algorithmic tangent operators coincide, since then $\lambda_A = \lambda_{A^*}$, and $D^{ep}_{AB} = \tilde{D}^{ep}_{AB} = D_{AB}$.

2.4.1. Outline of the numerical implementation

Following the above developments, we give a summary of the essential steps for the numerical implementation of multiplicative elasto-plasticity in the reference configuration. For simplicity we consider the case of perfect plasticity.

According to Section 2.4, the only input variables needed by the constitutive procedure are the total strains in form of $\boldsymbol{C}_{n+1}$ (or equivalently the Green–Lagrangian strains) in conjunction with the plastic history field $\boldsymbol{C}^{p-1}_n$.

Step 1: *Trial state*. Compute the eigenvalues $\lambda^2_{A^*}$ and the associated right eigenvectors $\boldsymbol{N}^A_*$ of $C^{p-1}_n C_{n+1}$ corresponding to the symmetric-definite generalized eigenvalue problem (37), i.e.

$$\left[\boldsymbol{C}_{n+1} - \lambda^2_{A^*}\boldsymbol{C}^p_n\right]\boldsymbol{N}^A_* = 0.$$

We refer to Golub and Van Loan (1996), section 8.7, for a description of efficient procedures for the solution of the underlying eigenvalue problem.

Step 2: *Check if the load step is elastic or plastic*. Compute the principal trial stress $\tau_{A^*} = \lambda_{A^*}\partial_{\lambda_A}\Psi(\lambda_{A^*})$ and check the yield condition. If $\Phi(\tau_{A^*}) \geqslant 0$ perform standard return mapping algorithm in principal stretches, based on eqn (41) see e.g. (Simo, 1992) to obtain λ_{An+1}. Compute second Piola–Kirchhoff stress tensor, which according to eqn (20) is given by

$$\boldsymbol{S}_{n+1} = \sum_{A=1}^{3} \tau_{An+1}\boldsymbol{N}^A_* \otimes \boldsymbol{N}^A_*, \quad \text{with } \tau_{An+1} = \lambda_{An+1}\partial_{\lambda_A}\Psi(\lambda_{An+1}).$$

Step 3: *Update the plastic history field*. According to eqn (13) the update of the plastic history field can be performed via

$$\boldsymbol{C}^{p-1}_{n+1} = \sum_{A=1}^{3} \lambda^2_{An+1}\boldsymbol{N}^A_* \otimes \boldsymbol{N}^A_*.$$

Step 4: *Algorithmic tangent operator*. Following the detailed derivation in the above, the fourth-order algorithmic elasto-plastic tangent tensor takes a particularly simple form when written as

$$\tilde{\mathbb{L}} = \tilde{L}_{ijkl}\boldsymbol{N}^i_* \otimes \boldsymbol{N}^j_* \otimes \boldsymbol{N}^k_* \otimes \boldsymbol{N}^l_*,$$

with components $\tilde{L}_{ijkl}$ according to eqn (46). Note that only nine independent components $\tilde{L}_{ijkl}$ have to be computed, which facilitates an efficient numerical implementation.

Note that the deformation gradient does not need to be provided by the finite element formulation which makes the considered constitutive procedure especially attractive for assumed strain elements.

2.5. Isotropic elasticity

Large-strain isotropic elasticity is contained as a special case in the more general setting considered above. In this section we briefly show that for elasticity our formulation degenerates to the formulas due to Bowen and Wang (1970), Chadwick and Ogden (1971) and Ogden (1984).

In the case of elasticity, the generalized eigenvalue problem (1) reduces to the standard symmetric form

$$\left[\boldsymbol{C} - \lambda_A^2 \boldsymbol{G}\right]\boldsymbol{N}^A = \boldsymbol{0}, \tag{47}$$

with $\boldsymbol{C}^p$ being replaced by the metric tensor $\boldsymbol{G}$ in the reference configuration. Accordingly, for distinct eigenvalues, we obtain three mutually orthogonal eigenvectors, such that, for $A \neq B, \boldsymbol{N}^A \cdot N^B = 0$, as well as $\boldsymbol{N}_A \cdot N_B = 0$. Next, we introduce an orthonormal set of eigenvectors defined by

$$\boldsymbol{n}_A = \frac{1}{\lambda_A}\boldsymbol{N}_A = \lambda_A \boldsymbol{N}^A. \tag{48}$$

Thus, the normalizing condition (10) yields $\boldsymbol{n}_A \cdot \boldsymbol{C}\boldsymbol{n}_A = \lambda_A^2$, and the orthogonality property (11) now takes the form

$$\boldsymbol{n}_A \cdot \boldsymbol{n}_B = \delta_{AB}. \tag{49}$$

Furthermore, the representations of $\boldsymbol{C}$ and $\boldsymbol{C}^p$ in eqn (12) now lead to the spectral decompositions

$$\boldsymbol{C} = \sum_{A=1}^{3} \lambda_A^2 \boldsymbol{n}_A \otimes \boldsymbol{n}_A \quad \text{and} \quad \boldsymbol{G} = \sum_{A=1}^{3} \boldsymbol{n}_A \otimes \boldsymbol{n}_A. \tag{50}$$

Moreover, expression (20) for the second Piola–Kirchhoff stress tensor may now be written in the form

$$\boldsymbol{S} = \sum_{A=1}^{3} \boldsymbol{S}_A \boldsymbol{N}_A \otimes \boldsymbol{n}_A, \quad \text{with} \quad S_A = \frac{\tau_A}{\lambda_A^2} = \frac{\partial_{\lambda_A}\Psi}{\lambda_A}. \tag{51}$$

Here, the components S_A are the principal second Piola–Kirchhoff stresses. The elastic tangent operator corresponding to eqn (28) can now be written as

$$\mathbb{L} = \Lambda_{ijkl} \boldsymbol{n}_i \otimes \boldsymbol{n}_j \otimes \boldsymbol{n}_k \otimes \boldsymbol{n}_l, \tag{52}$$

where the components Λ_{ijkl} follow from eqn (34) and (35) via

$$\Lambda_{ABCD} = \frac{L_{ABCD}}{\lambda_A \lambda_B \lambda_C \lambda_D}. \tag{53}$$

Accordingly, for the case of elasticity, we obtain the familiar expressions

$$\Lambda_{AABB} = \partial_{e_B} S_A \quad A, B = 1, 2, 3,$$

$$\Lambda_{ABAB} = \frac{S_B - S_A}{2[e_B - e_A]} \quad A \neq B. \tag{54}$$

Here, $e_A = \frac{1}{2}[\lambda_A^2 - 1]$ are the principal Green–Lagrangian strains. Eqn $(54)_1$ follows from eqn (34) by taking into account that $\partial_{e_A}(\bullet) = \lambda_A^{-1}\partial_{\lambda_A}(\bullet)$, such that

$$\partial_{e_B} S_A = \frac{D_{AB}}{\lambda_A^2 \lambda_B^2} - \frac{2S_A}{\lambda_A^2} \delta_{AB}, \tag{55}$$

where the components D_{AB} are given by eqn (A2). The limiting case of multiple eigenvalues can be treated with the help of eqn (36), which with regard to eqns (53) and (55) leads to

$$\Lambda_{ABAB} = \tfrac{1}{2} \partial_{e_B} [S_B - S_A], \quad \text{if } e_A = e_B, \quad A \neq B. \tag{56}$$

3. Conclusions

We have derived the continuous and algorithmic form of the fourth-order tangent operator corresponding to multiplicative elasto-plasticity formulated within a material setting. To this end we extended an approach due to Bowen and Wang (1970) and Chadwick and Ogden (1971), originally developed for the case of finite elasticity in principal axis. In the present more general framework of large strain elasto-plasticity, we start from a generalized eigenvalue problem which belongs to the class of symmetric-definite eigenproblems. Due to its special structure, the generalized eigenvalue problem facilitates certain tensor representations in terms of a reciprocal set of left and right eigenvectors. These representations replace the spectral decompositions of the original approach based on an orthonormal set of eigenvectors.

It turns out that the conceptual simplicity and elegance of the original formulation carries over to the more general context of multiplicative elasto-plasticity. Specifically, the resolution of the tangent operator in the basis spanned by the right eigenvectors results in extremely simple component expressions. As a further result it is shown that the algorithmic version of the tangent operator preserves entirely the structure of the continuous form. For example, similar to the infinitesimal theory of elasto-plasticity (Simo and Taylor, 1985), the continuous elasto-plastic moduli formulated in principle axis are replaced by their algorithmic counterpart consistent with the integration of the flow rule also written in principal axis. Naturally, our generalized formulation includes finite elasticity as a special case. In this case the tangent operator reduces to the familiar form of the original formulation.

Even higher order derivatives such as the sixth-order tensor corresponding to the second gradient of $\boldsymbol{S}$ with respect to $\boldsymbol{C}$ could be easily calculated by employing the newly developed generalization to elasto-plasticity. This might be of interest for the investigation of stability problems (Reese and Wriggers, 1995) where the original approach restricted to elasticity has been used.

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Appendix A. Continuous elasto-plastic moduli D_{AB}^{ep}

For simplicity we consider the case of perfect plasticity with arbitrary stored energy function $\Psi = \Psi(\lambda_1, \lambda_2, \lambda_3)$ and yield function $\Phi = \Phi(\tau_1, \tau_2, \tau_3)$. Since $\tau_A = \lambda_A \partial_{\lambda_A} \Psi$, time differentiation yields $\dot{\tau}_A = \partial_{\lambda_i} \tau_A \dot{\lambda}_i$. Employing the flow rule in the form of eqn (27), i.e. $\dot{\lambda}_B = \lambda_B [\frac{1}{2}(\dot{\boldsymbol{C}}^{BB}) - \dot{\gamma} \partial_{\tau_B} \Phi]$, yields

$$\dot{\tau}_A = \sum_{B=1}^{3} D_{AB}\left[\frac{1}{2}(\dot{\boldsymbol{C}})^{BB} - \dot{\gamma}\,\partial_{\tau_B}\Phi\right], \tag{A1}$$

with

$$D_{AB} = \tau_A \delta_{AB} + \lambda_A \partial^2_{\lambda_A \lambda_B} \Psi \lambda_B. \tag{A2}$$

Note that eqn (A2) may also be written in the form $D_{AB} = \partial^2_{\varepsilon_A \varepsilon_B}\psi$, where the logarithmic stretches $\varepsilon_A = \ln \lambda_A$ have been introduced. For plastic loading, the consistency condition $\dot{\Phi} = 0$ leads to $\dot{\Phi} = \partial_{\tau_i}\Phi\dot{\tau}_i = \partial_{\tau_i}\Phi D_{ij}[\frac{1}{2}(\dot{\boldsymbol{C}})^{ij} - \dot{\gamma}\,\partial_{\tau_j}\Phi] = 0$. Thus, the plastic parameter takes the form

$$\dot{\gamma} = \frac{\partial_{\tau_i}\Phi D_{ij}(\dot{\boldsymbol{C}})^{(jj)}}{2\partial_{\tau_k}\Phi D_{kl}\partial_{\tau_l}\Phi}. \tag{A3}$$

Eventually, (A1) may be written in the form

$$\dot{\tau}_A = \sum_{B=1}^{3} D^{ep}_{AB}\frac{1}{2}(\dot{\boldsymbol{C}})^{BB}, \tag{A4}$$

where the continuous elasto-plastic moduli are given by

$$D^{ep}_{AB} = D_{AB} - \frac{\partial_{\tau_i}\Phi D_{Ai}\partial_{\tau_j}\Phi D_{Bj}}{\partial_{\tau_k}\Phi D_{kl}\partial_{\tau_l}\Phi}. \tag{A5}$$

Appendix B. Algorithmic elasto-plastic moduli $\tilde{D}^{ep}_{AB}$

Similarly to the derivation of the continuous elasto-plastic moduli, the algorithmic elasto-plastic moduli, consistent with the integrated version of the flow rule, can be calculated. Using the algorithmic flow rule in the form of eqn (41), i.e. $\varepsilon_A = \varepsilon_{A^*} - \gamma\partial_{\tau_A}, \Phi$, in combination with $\tau_A = \partial_{\varepsilon_A}\Psi$ yields

$$\Delta\tau_A = \sum_{B=1}^{3} H_{AB}\left[\Delta\varepsilon_{B^*} - \Delta\gamma\partial_{\tau_B}\Phi\right], \tag{B1}$$

where the matrix $\boldsymbol{H} = [H_{AB}]$ is defined by

$$\boldsymbol{H}^{-1} = \boldsymbol{D}^{-1} + \gamma\left[\partial^2_{\tau_A\tau_B}\Phi\right], \tag{B2}$$

where $\boldsymbol{D} = [D_{AB}]$, with components D_{AB} in eqn (A2). The consistency condition implies $\Delta\Phi = \partial_{\tau_i}\Phi\Delta\tau_i = 0$, from which the linearized plastic parameter follows in the form

$$\Delta\gamma = \frac{\partial_{\tau_i}\Phi H_{ij}\Delta\varepsilon_{j^*}}{\partial_{\tau_k}\Phi H_{kl}\partial_{\tau_l}\Phi}. \tag{B3}$$

Substituting eqn (B3) into eqn (B1) leads to

$$\Delta\tau_A = \sum_{B=1}^{3} \tilde{D}^{ep}_{AB}\Delta\varepsilon_{B^*}, \tag{B4}$$

where the algorithmic elasto-plastic moduli are given by

$$\tilde{D}^{ep}_{AB} = H_{AB} - \frac{\partial_{\tau_i}\Phi H_{Ai}\,\partial_{\tau_j}\Phi H_{Bj}}{\partial_{\tau_k}\Phi H_{kl}\partial_{\tau_l}\Phi}. \tag{B5}$$

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